

# Feature Propagation on Graph: A New Perspective to Graph Representation Learning

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## Abstract

We study feature propagation on graph, an inference process involved in graph representation learning tasks. It’s to spread the features over the whole graph to the  $t$ -th orders, thus to expand the end’s features. The process has been successfully adopted in graph embedding or graph neural networks, however few works studied the convergence of feature propagation. Without convergence guarantees, it may lead to unexpected numerical overflows and task failures. In this paper, we first define the concept of feature propagation on graph formally, and then study its convergence conditions to equilibrium states. We further link feature propagation to several established approaches such as node2vec and structure2vec. In the end of this paper, we extend existing approaches from represent nodes to edges (edge2vec) and demonstrate its applications on fraud transaction detection in real world scenario. Experiments show that it is quite competitive.

## 1 Introduction

In this paper, we study the feature propagation on graph, which forms the building blocks in many graph representation learning tasks. Typically, the graph representation learning tasks aim to learn a function  $f(\mathcal{X}, \mathcal{G})$  to somehow utilize the additional graph structure in space  $\mathcal{G}$ , compared with traditional learning tasks  $f(\mathcal{X})$  by only considering each sample independently. The successes of graph representation approaches [Grover and Leskovec, 2016; Dai *et al.*, 2016; Kipf and Welling, 2016; Hamilton *et al.*, 2017] have proven to be successful on citation networks [Sen *et al.*, 2008], biological networks [Zitnik and Leskovec, 2017], and transaction networks [Liu *et al.*, 2017] that can be formulated in graph structures.

One major process of graph representation learning tasks involves the feature propagation over the graph up to  $t$ -th orders. Those approaches define various propagation manners based on such as, adjacency matrices [Belkin and Niyogi, 2002],  $t$ -order adjacency matrices [Cao *et al.*, 2015], expected co-occurrence matrices [Perozzi *et al.*, 2014][Grover and Leskovec, 2016] by conducting random walks. Recently,

graph convolutional networks have shown their promising results on various datasets. They rely on either graph Laplacians [Kipf and Welling, 2016] or on carefully-designed operators like mean, max operators over adjacency matrix [Hamilton *et al.*, 2017].

However, few of graph representation learning tasks study the propagation process used in their inference procedures. For instance, GCN [Kipf and Welling, 2016] or structure2vec [Dai *et al.*, 2016] implicitly involve this procedure in the form

$$H^{(t+1)} = \phi(A)H^{(t)}W,$$

where  $H \in \mathbb{R}^{N,K}$  denotes the learned embeddings of  $N$  nodes in vector space  $\mathbb{R}^K$ , the  $t$  denotes the  $t$ -th iteration,  $\phi(\cdot)$  defines the operator on adjacency matrix  $A \in \{0, 1\}^{N,N}$  given graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ . This propagation process is parameterized by  $W \in \mathbb{R}^{K,K}$ . This iterative propagation process essentially propagate and spread each node  $i$ ’s signals to  $i$ ’s  $T$ -th step neighborhood over the graph. Without the careful designs of the process under certain conditions, the propagation could be under risk of numeric issues.

In this paper, we are interested in the convergence condition of the propagation process to equilibrium state [Langville and Meyer, 2006], hopefully can help the understanding of existing literatures in this domain: (1) we first formulate the generic framework of feature propagation on graphs; (2) we connect existing classic approaches such as node2vec [Grover and Leskovec, 2016], a random walk based graph embedding approach, and structure2vec [Dai *et al.*, 2016], a graph convolution based approach, to our feature propagation framework; (3) we study the convergence condition of feature propagation over graph to equilibrium state with  $T \rightarrow \infty$  by using theory of M-matrix [Plemmons, 1977], which is quite simple and easy to implement by gradient projection; (4) we further extend the existing node representation approaches to edge representation, i.e. we propose “edge2vec” and show its applications on fraud transaction detection in a real world transaction networks, which is essentially important in any financial systems. More importantly, “edge2vec” can deal with multiple links (transaction among two accounts over a time period) among two nodes, which is essentially different from traditional settings like recommender systems (the user  $i$  could have only one rating  $r_{ij}$  on the item  $j$ , i.e. only one link among two nodes).

This paper is organized as follows. In section 2, we sets

up the preliminary of this paper, and propose pairs of general definitions for feature expansion and feature propagation in a unified learning framework. In section 3, we discuss a typical feature propagation way, and propose the sufficient conditions for its convergence. In section 4, we explore the connection between feature propagation and two types of graph representation approaches. We finally extend the node embedding to edge embedding, and demonstrated its effectiveness by conducting experiments on fraud transaction detection in section 5 and section 6 respectively.

## 2 Preliminary

Suppose the graph is  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V} = \{1, 2, \dots, n\}$  is the node set,  $\mathcal{E} = \{e_1, e_2, \dots, e_m\}$  is the edge set, and the adjacency matrix is  $A = [a_{ij}]_{n \times n}$  where  $a_{ij} = 1$  if  $(i, j) \in \mathcal{E}$ , and 0 otherwise.  $D = \text{diag}\{d_1, d_2, \dots, d_n\}$  is the degree matrix of graph where  $d_i = \sum_{j=1}^n a_{ij}$  is the degree of node  $i$  (or  $[d_1, d_2, \dots, d_n] = Ae$ ). The feature set for node is  $X = \{x_1, x_2, \dots, x_n\}$  where  $x_i = [x_{i1}, x_{i2}, \dots, x_{id}]^T$  responds to node  $v_i$ , and denotes its responding feature matrix as  $X = [x_1, x_2, \dots, x_n]^T$ . If exists, we denote the feature set for edge as  $X^{(e)} = \{x_1^{(e)}, x_2^{(e)}, \dots, x_m^{(e)}\}$  where  $x_i^{(e)} = [x_{i1}^{(e)}, x_{i2}^{(e)}, \dots, x_{id(e)}^{(e)}]^T$  responds to the edge  $e_i$  and denote its responding feature matrix as  $X^{(e)} = [x_1^{(e)}, x_2^{(e)}, \dots, x_m^{(e)}]^T$ . If the label locates in node, we denote the label vector as  $Y = [y_1, y_2, \dots, y_n]^T$ , and if the label locates in edge, we denote the label vector as  $Y^{(e)} = [y_1^{(e)}, y_2^{(e)}, \dots, y_m^{(e)}]^T$ .

For the traditional learning tasks (with or without graph topology), the typical way to build the fitting model is as follows

$$Y = f(X; \theta) \quad \text{or} \quad Y^{(e)} = f(X^{(e)}; \theta).$$

However, this way only utilizes the features of node or edge itself. In a context-aware perspective, the features of neighbor or the neighbor's neighbor may also be useful. For example, in a social network, assume that one person didn't fill her age, it may be hard to get this feature once we only utilize the features of herself; but if we utilize her neighbors' features, we may estimate this feature by averaging her neighbors' ages or take their median. We denote the expanded feature as  $\widetilde{X}$  and  $\widetilde{X}^{(e)}$  according to the raw feature  $X$  and  $X^{(e)}$  respectively. And call the expanded process from  $X$  and  $X^{(e)}$  to  $\widetilde{X}$  and  $\widetilde{X}^{(e)}$  as feature expansion. We define this concept as follows

**Definition 1 (Feature Expansion).** Suppose the raw feature of graph are  $X$  and  $X^{(e)}$ , responding to node and edge respectively, if

$$\widetilde{X} = \mathcal{P}(X, X^{(e)}; \theta_p) \quad \text{or} \quad \widetilde{X}^{(e)} = \mathcal{P}(X^{(e)}, X; \theta_p)$$

then we call  $\widetilde{X}$  or  $\widetilde{X}^{(e)}$  as expanded features, and call the function  $\mathcal{P}$  as feature expansion function.

With expanded features, the fitting model will be

$$\begin{aligned} Y &= f(\widetilde{X}; \theta) = f(\mathcal{P}(X, X^{(e)}; \theta_p); \theta) \quad \text{or} \\ Y^{(e)} &= f(\widetilde{X}^{(e)}; \theta) = f(\mathcal{P}(X^{(e)}, X; \theta_p); \theta) \end{aligned} \quad (1)$$

which contains two sets of parameters  $\theta_p$  and  $\theta$ , where  $\theta_p$  is parameters for feature expansion and  $\theta$  is for fitting the final

label. And the learning framework with feature expansion is as follows

1. Initialize parameters  $\theta_p$  and  $\theta$ ;
2. Expand the raw feature  $X$  to  $\widetilde{X}$  by expansion function  $\mathcal{P}(X, X^{(e)}; \theta_p)$ ;
3. Compute the prediction  $\hat{Y} = f(\widetilde{X}; \theta)$ ;
4. Back propagate the  $\text{loss}(Y, \hat{Y})$  to update  $\theta$  and  $\theta_p$ ;
5. Repeat step 2-4 until  $\text{loss}(Y, \hat{Y})$  minimized;

In graph, the feature expansion is usually propagated via the graph topology, and the feature of node or edge is expanded by its neighbors in the  $t$ -th orders. Since this feature expansion process relies on the feature propagation through the graph topology, we call this process as feature propagation with definition as follows:

**Definition 2 (Feature Propagation).** Suppose the raw feature of graph are  $X = [x_1, x_2, \dots, x_n]^T$  and  $X^{(e)} = [x_1^{(e)}, x_2^{(e)}, \dots, x_m^{(e)}]^T$ , responding to node and edge respectively, if for each  $i \in [1, n]$  and  $j \in [1, m]$

$$\begin{aligned} \widetilde{x}_i &= \mathcal{P}(x_i, \quad \{ \widetilde{x}_k \}_{k \text{ is } i's \text{ neighbor}}, \quad \{ x_k^{(e)} \}_{e_k \text{ is adjacent to node } i}; \theta_p) \quad \text{or} \\ \widetilde{x}_j^{(e)} &= \mathcal{P}(x_j^{(e)}, \quad \{ \widetilde{x}_k \}_{\text{node } k \text{ is related to } e_j}; \theta_p) \end{aligned} \quad (2)$$

, then we call  $\widetilde{X}$  or  $\widetilde{X}^{(e)}$  as propagation-expanded feature, and call the function  $\mathcal{P}$  as feature propagation function.

Although in feature propagation each node/edge only takes advantage of its neighbors' information, it still could get the information farther away through the iteratively propagation of definition 2.

In this section, we propose the general definitions for feature expansion and feature propagation in graph and propose the learning framework with feature expansion. In the next section, we will discuss a typical feature propagation way, which has strong connection with the recent popular graph representation learning method.

## 3 A Typical Way for Feature Propagation

The typical way to expand node's features by propagation is as follows, which is a generalization of pagerank equation [Page *et al.*, 1999; Xiang *et al.*, 2013],

$$\widetilde{x}_i = W_1^T x_i + W_2^T \sum_{j \in \mathcal{N}(i)} \widetilde{x}_j \quad \text{for } i = 1, 2, \dots, n \quad (3)$$

where  $\mathcal{N}(i)$  is the neighbor set of node  $i$ ,  $W_1 = [w_{ij}^{(1)}]_{d \times d'}$  and  $W_2 = [w_{ij}^{(2)}]_{d' \times d'}$  are the parameters of Eq. 3 (thus, the dimension of  $\widetilde{x}_i$  is  $d'$ ). For the convenience, we call  $W_2$  as propagation matrix in this paper. This equation group could be rewritten as

$$\widetilde{X} = XW_1 + A\widetilde{X}W_2 \quad (4)$$

Breaking up the group of equations, we have

$$\begin{aligned} \widetilde{x}_i &= \sum_{k=1}^d x_{ik} w_{kj}^{(1)} + \sum_{p=1}^n \sum_{q=1}^{d'} a_{ip} \widetilde{x}_{pq} w_{qj}^{(2)} \\ &\quad \text{for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, d' \end{aligned} \quad (5)$$

And let  $s = i * n + j$ ,  $Z = [z_s]_{1*(n*d')}$  with  $z_s = \widetilde{x}_{ij}$ ,  $L = [l_s]_{1*(n*d')}$  with  $l_s = \sum_{k=1}^d x_{ik} w_{kj}^{(1)}$  and  $A' = [a'_{st}]_{(n*d')*(n*d')}$  with

$$a'_{st} = a_{ip} w_{qj}^{(2)} = \begin{cases} 0 & \text{if } a_{ip} = 0 \\ w_{qj}^{(2)} & \text{otherwise} \end{cases}, \quad \text{with} \quad (6)$$

$$s = i * n + j, t = p * n + q$$

Equation 5 could be rewritten as

$$z_s = l_s + \sum_{t=1}^{n*d'} z_t * a'_{st} \quad \text{for } s = 1, 2, \dots, n * d'$$

After summing up, it becomes

$$Z = L + A'Z$$

If matrix  $(I - A')$  is invertible, we will get

$$Z = (I - A')^{-1}L \quad (7)$$

However,  $(I - A')$  is not invertible naturally, we should set some conditions to make it be. From Equation 6, we could see that, only propagation matrix  $W_2$  will affect the invertibility of  $(I - A')$ .

From the theory of M-matrix, if  $(I - A')$  satisfies the following two conditions, it will be invertible.

1.  $A' \geq 0$ , which, by Eq. 6, is equivalent to  $W_2$  should be a nonnegative matrix;
2.  $A'\mathbf{e} < \mathbf{e}$ , which, with the derivation in footnote<sup>1</sup>, is equivalent to  $\max\{W_2^T \mathbf{e}\} < 1/\max\{d_1, d_2, \dots, d_n\}$

However, condition 2 is a very demanding condition. If there exists a node with very large degree, the row sum of  $W_2$  will have to be very small. To solve this issue, we could make the below changes to Eq. 4, i.e. replace the matrix  $A$  as  $D^{-1}A$ . Then, Eq. 4 changes to

$$\widetilde{X} = XW_1 + D^{-1}A\widetilde{X}W_2 \quad (8)$$

Dive into each  $\widetilde{x}_i$ , we have

$$\widetilde{x}_i = W_1^T x_i + W_2^T \sum_{j \in N(i)} \frac{1}{d_i} \widetilde{x}_j \quad (9)$$

Under this feature propagation process, the above condition 2 will change to  $\max\{W_2 \mathbf{e}\} \leq 1$ . Comparatively, this condition is easier to be guaranteed.

Summing up the above derivations, we form the following theorem.

**Theorem 1** *For feature propagation method as Eq. 8 or 9, the propagation matrix  $W_2$  if satisfy the following conditions, the propagation process will be convergent.*

<sup>1</sup>let's denote  $W_2^T \mathbf{e} = [w_1, w_2, \dots, w_{d'}]^T$ , from Eq. 6,

$$\begin{aligned} A'\mathbf{e} < \mathbf{e} &\iff \forall s, \sum_{t=1}^{n*d'} a'_{st} < 1 \iff \forall i, j, \sum_{p=1}^n \sum_{q=1}^{d'} a_{ip} w_{qj}^{(2)} < 1 \\ &\iff \forall i, j, \sum_{p=1}^n a_{ip} \sum_{q=1}^{d'} w_{qj}^{(2)} < 1 \iff \forall i, j, d_i w_j < 1 \\ &\iff \max\{w_1, w_2, \dots, w_{d'}\} < 1/\max\{d_1, d_2, \dots, d_n\} \end{aligned}$$

- condition 1.  $W_2$  is nonnegative.
- condition 2.  $\max\{W_2^T \mathbf{e}\} < 1$ .

Theorem 1 proposed a pair of sufficient conditions to guarantee the convergence of feature propagation, but they are not necessary conditions. When the propagation matrix  $W_2$  satisfies the conditions in theorem 1, the feature propagation process as Eq. 8 will be convergent. Otherwise, the feature expansion may lead to explode which actually has been confirmed by the practical experiences.

## 4 Relationship to Graph Representation Learning

Recent years have seen a surge of research on graph representation and node embedding. These works could be roughly categorized into two types: 1) embeddings with graph structure only [Perozzi *et al.*, 2014; Grover and Leskovec, 2016; Abu-El-Haija *et al.*, 2017], and 2) embeddings with both structure and features (or attributes) [Kipf and Welling, 2016; Dai *et al.*, 2016; Hamilton *et al.*, 2017]. In this section, we discuss the relationship between feature propagation and graph representation.

### 4.1 With Graph Structure Only

For the typical feature propagation way as Eq. 8, if we let each node feature  $x_i$  as a one-hot vector<sup>2</sup> (i.e.  $X = I$ ),  $W_1$  as a randomly initialized matrix  $C = [c_1, c_2, \dots, c_{d'}] = [c_{ij}]_{n*d'}$ ,  $W_2 = \alpha I$  ( $\alpha < 1$ ) (must satisfy the two conditions in Theorem 1), and denote  $T = D^{-1}A$ <sup>3</sup>, then Eq. 8 will be

$$\widetilde{X} = C + \alpha T \widetilde{X}. \quad (10)$$

If substituting the above equation into its left side recursively, we will get

$$\widetilde{X} = (I + \alpha T + \alpha^2 T^2 + \dots)C. \quad (11)$$

Let's denote

$$P = [p_1, p_2, \dots, p_n]^T = [p_{ij}]_{n*n} = \lim_{k \rightarrow \infty} \sum_k \alpha^k T^k.$$

Because  $\alpha < 1$ , the infinite sequence of  $P$  will be converged gradually. Approximately,  $T^k$  is the  $k$ -step transition probability matrix between any pair of nodes. Thus,  $P$  is the weighted sum of  $k$ -step transition matrix with weight  $\alpha^k$  and we call  $P$  as proximity matrix. Its entry  $p_{ij}$  depicts the transition probability from node  $i$  to node  $j$  by 0-step, 1-step, up to  $\infty$ -steps, and  $p_i$  depicts the transition probability from node  $i$  to any node in the graph. Thus, if node  $i$  and node  $j$  close to each other in the graph,  $p_i$  and  $p_j$  will be close too. From Eq. 11, we have

$$\widetilde{x}_i = [p_i c_1, p_i c_2, \dots, p_i c_{d'}],$$

then

$$\widetilde{x}_i - \widetilde{x}_j = [(p_i - p_j)c_1, (p_i - p_j)c_2, \dots, (p_i - p_j)c_{d'}].$$

<sup>2</sup>which means node  $i$  contains no feature, but only its identity.

<sup>3</sup>traditionally,  $T$  is called as transition matrix, there is  $T\mathbf{e} = \mathbf{e}$

If node  $i$  is close to node  $j$  in graph (which means  $p_i$  is close to  $p_j$ <sup>4</sup>), then  $\tilde{x}_i - \tilde{x}_j$  will close to 0 no matter how the  $C$  is initialized. In [Abu-El-Haija *et al.*, 2017], the authors revisited DeepWalk [Perozzi *et al.*, 2014] and GloVe [Pennington *et al.*, 2014], and find that their proximity matrices are:

$$P^{\text{DeepWalk}[K]} = \sum_{k=1}^K \left[1 - \frac{k-1}{K}\right] T^k, \quad (12)$$

$$P^{\text{GloVe}[K]} = \sum_{k=1}^K \left[\frac{1}{k}\right] T^k,$$

respectively. Compared with the two proximity matrices above, the major differences between ours  $P$  and theirs is the decay weight of  $T^k$ . And our weight  $\alpha^k$  is as reasonable as  $1 - (k-1)/K$  or  $1/k$ . Thus,  $\tilde{X}$  is a reasonable first type embedding.

Figure 1 displays an example embedding for the famous Zachary Karate Club social network [Perozzi *et al.*, 2014], where we use two dimensional node embeddings to capture the community structure implicit in the social network. We changed the initialization of  $W_1$  and  $\alpha$  in  $W_2$ , and could see that:

1. no matter how the  $W_1$  is initialized, the embeddings can capture the community structures in the network pretty well;
2. as the propagate parameter  $\alpha$  becomes larger, the nodes in a community will tend to aggregate;

Feature propagation as Eq. 8 could be a simple way of the first type embedding when  $W_1$  and  $W_2$  satisfy center conditions.

## 4.2 With Structure and Features on Graph Simultaneously

In structure2vec [Dai *et al.*, 2016], a graph convolution based approach, the node embedding was formulated as

$$\tilde{\mu}_i = \sigma(W_1 x_i + W_2 \sum_{j \in N(i)} \tilde{\mu}_j + W_3 \sum_{j \in N(i)} x_j)$$

where  $\sigma := \max\{0, \cdot\}$  is a rectified linear unit function. Suppose the dimension of  $\tilde{\mu}_i$  is  $d'$ , i.e.  $\tilde{\mu}_i = [\tilde{\mu}_{i1}, \tilde{\mu}_{i2}, \dots, \tilde{\mu}_{id'}]$ . Using the similar derivations in section 3, we can get

$$\tilde{\mu}_{ij} = \sigma\left(c_{ij} + \sum_{p=1}^n \sum_{q=1}^{d'} a_{ip} \tilde{\mu}_{pq} W_{qj}^{(2)}\right) \quad (13)$$

for  $i = 1, 2, \dots, n$ , and  $j = 1, 2, \dots, d'$

where  $c_{ij} = \sum_{k=1}^d w_{jk}^{(1)} x_{ik} + \sum_{p=1}^n \sum_{q=1}^{d'} a_{ip} x_{pq} w_{qj}^{(3)}$ . Without loss of generality, let's suppose there are  $K$  variables  $[\tilde{\mu}_{i_1 j_1}, \tilde{\mu}_{i_2 j_2}, \dots, \tilde{\mu}_{i_K j_K}] \equiv \mu$  in Eq. (13) are nonzeros, while the other  $n * d' - K$  variables are equal to 0s. Then, Eq. (13)

<sup>4</sup>for  $p_{ii}$  must be larger than  $p_{ji}$  and  $p_{jj}$  must be larger than  $p_{ij}$ , it will impact the comparison between  $p_i$  and  $p_j$ . The better way is to adjust  $P$  as  $\hat{P} = P - I$  or adjust  $\tilde{X}$  as  $\hat{X} = \hat{P}C = (\alpha T + \alpha^2 T^2 + \dots)C = \tilde{X} - C$ .

could be rewritten as

$$\tilde{\mu}_{i_s j_s} = c_{ij} + \sum_{p=1}^n \sum_{q=1}^{d'} a_{i_s p} \tilde{\mu}_{pq} W_{q j_s}^{(2)} \quad (14)$$

for  $s = 1, 2, \dots, K$

This equation also could be resolved by the similar derivations in section 3. The final solution is in the form of

$$\mu = (I - A'')^{-1} C$$

where  $C = [c_{ij}]$  and  $A''$  is the matrix  $A'$  in section 3 after removing  $n * d' - K$  corresponding rows and columns.

Similarly, if we want the node embeddings converge and do not get explode, the matrix  $W_2$  also needs to satisfy certain conditions like in section 3. The relu function  $\sigma$  decreased the scale of equations, but it hasn't changed the essence of linear system.

## 5 Extension to Edge

The above section discussed the feature propagation when the graph only contains node features(i.e.  $X$ ). However, in the many real scenarios, the graph may contains edge features(i.e.  $X^{(e)}$ ) too. If we neglect the edge features, it may weaken the model's performance. What's more, the label may locate in edge directly, we have to utilize the edge features especially when there exists multiple links between two nodes. This section we will discuss the feature propagation when the graph contains edge features in multiple-links settings.

For each edge  $e_i$ , suppose  $i_s$  and  $i_t$  are source node and target node of  $e_i$  respectively, in mathematical form, i.e.  $e_i = (i_s, i_t)$ . Suppose  $\mathcal{S}(k)$  is the set of edge which takes node  $k$  as source node and  $\mathcal{T}(k)$  is the set of edge which takes node  $k$  as target node. Suppose

$$C_s = [c_{ij}^{(s)}]_{m * n}, \text{ where } c_{ij}^{(s)} = \begin{cases} 1 & \text{if } e_i \in \mathcal{S}(j) \\ 0 & \text{otherwise} \end{cases} \quad (15)$$

$$C_t = [c_{ij}^{(t)}]_{m * n}, \text{ where } c_{ij}^{(t)} = \begin{cases} 1 & \text{if } e_i \in \mathcal{T}(j) \\ 0 & \text{otherwise} \end{cases}$$

and we call  $C_s$  and  $C_t$  as source incidence matrix and target incidence matrix respectively. Obviously, there is

$$C_s^T \cdot C_s = I, \quad C_s^T \cdot C_t = 0$$

$$C_t^T \cdot C_s = 0, \quad C_t^T \cdot C_t = I \quad (16)$$

We could expand the features by the following way

$$\begin{cases} \tilde{x}_i^{(e)} &= W_1^T x_i^{(e)} + W_2^T \tilde{x}_{i_s} + W_3^T \tilde{x}_{i_t} \\ \tilde{x}_j &= W_4^T x_j + W_5 \sum_{k \in N(j)} \frac{1}{d_j} \tilde{x}_k + W_6^T \sum_{e_s \in \mathcal{S}(j)} \tilde{x}_s^{(e)} + W_7^T \sum_{e_t \in \mathcal{T}(j)} \tilde{x}_t^{(e)} \end{cases} \quad (17)$$

which could be rewritten as

$$\begin{cases} \tilde{X}^{(e)} &= X^{(e)} W_1 + C_s \tilde{X} W_2 + C_t \tilde{X} W_3 \\ \tilde{X} &= X W_4 + D^{-1} A \tilde{X} W_5 + A C_s^T \tilde{X}^{(e)} W_6 + A C_t \tilde{X}^{(e)} W_7 \end{cases} \quad (18)$$

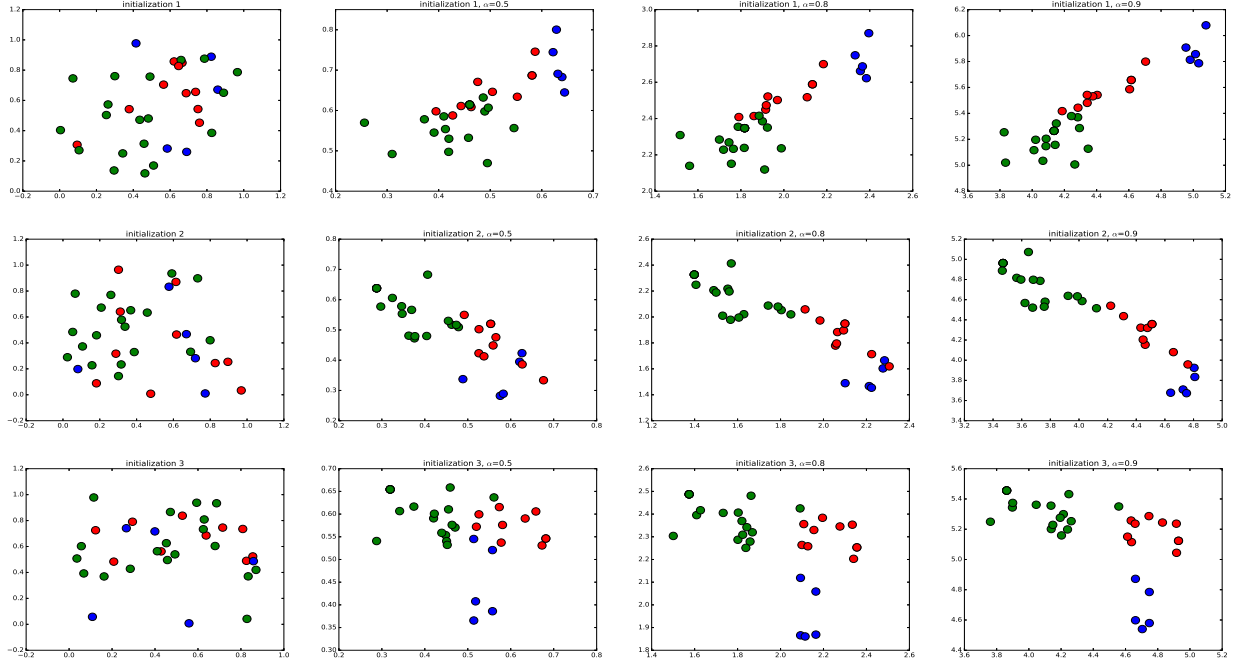


Figure 1: Node embedding for Zachary Karate Club social network. Different color indicates different community. Each row of figures is a group of results with the same initialization of  $W_1$  and different  $\alpha \in \{0.5, 0.8, 0.9\}$ .

combine Eq. 16 and Eq. 18, we will get

$$\begin{aligned} \widetilde{X} &= (XW_4 + AC_5^T X^{(e)} W_1 W_6 + AC_7^T X^{(e)} W_1 W_7) + \\ & D^{-1} A \widetilde{X} (W_5 + W_2 W_6 + W_2 W_7) \end{aligned} \quad (19)$$

Follow the similar derivations in section 3, we could know that if we want the feature propagation to be convergent, we should satisfy certain conditions. Based on theorem 1, the following two conditions can guarantee the convergence of the above feature propagation process:

1.  $(W_5 + W_2 W_6 + W_2 W_7)$  should be nonnegative;
2.  $\max\{(W_5 + W_2 W_6 + W_2 W_7)^T \mathbf{e}\} < 1$

The above condition 2 is not easy to be satisfied since it depends on the interaction between different matrices. And, the feature propagation with edge features on graph is easy to explode. To eliminate this obstacle, we could simplify the feature propagation process by setting  $\widetilde{W}_6$  and  $W_7$  to be 0, which means the expanded features  $\widetilde{X}$  on nodes won't rely on features of edges. Then, we have the following feature propagation equation:

$$\begin{cases} \widetilde{X}^{(e)} &= X^{(e)} W_1 + C_5 \widetilde{X} W_2 + C_7 \widetilde{X} W_3 \\ \widetilde{X} &= X W_4 + D^{-1} A \widetilde{X} W_5 \end{cases} \quad (20)$$

We call this feature propagation way for edge as edge2vec in this paper. For edge2vec, we only need to guarantee the following conditions

1.  $W_5 > 0$ ;
  2.  $\max\{W_5^T \mathbf{e}\} < 1$ ;
- to guarantee its convergence.

## 6 Applications in Fraud Transaction Detection

In the previous sections, we first proposed the concept ‘‘feature propagation’’ in a unified framework. We link feature propagation as a basic building block to several graph representation tasks, and point out that the convergence conditions involved in generic graph representation tasks. We further propose a simple extension of feature propagation to edge2vec where features and labels located on edges. In this section, we conduct experiments on real world data to demonstrate the performance of edge2vec and its convergence.

### 6.1 DataSet

In this section, we study a real world data at a leading cashless payment platform in the world, served more than hundred millions of users. As a financial services provider, one of major problems faced is the risk control of fraudulent transactions. Detecting and identifying the risk of fraud for each transaction plays the fundamental importance of the platform.

In particular, we study the fraud transaction in the online shopping setting, where sellers sell fake items to customers to reap undeserved profits. Independently considering each transaction between a seller and a buyer cannot characterize useful information from the whole transaction network. Considering the problem in the feature propagation framework over graph can help us understand underlying aggregation pattern of the fraudulent transactions.

The experimental fraud transaction data<sup>5</sup> contains three

<sup>5</sup>the data is randomly sampled over a time period with complete data desensitization (no personal profile, no user id).

types of features: 1) buyer’s features 2) seller’s features and 3) characterizations on each transaction. We treat each buyer and seller as a node of the graph, and each transaction is an edge between buyer and seller. If one transaction  $e_i$  is fraud, we label its corresponding edge as  $y_i^{(e)} = [1, 0]$ , otherwise label the transaction  $e_i$  as  $y_i^{(e)} = [0, 1]$ . Our task is to predict whether or not one edge is a fraud. The detailed statistics of the data is described in Table 2.

Note that there could be multiple edges between a seller and a buyer, thus make the setting a bit different from traditional recommendation setting. Our edge2vec can embed each edge into a vector space, so that it can help us to infer the risk of each edge in the graph.

Table 1: Fraud Transaction Detection Data Description.

	#Nodes	#Edges	#Fraud	#Normal
Training Data	626,003	1,720,180	31,737	1,688,441
Testing Data	1,355,824	4,034,962	86,721	3,948,241

## 6.2 Treatment and Control Groups

As discussed in section 2, the learning framework is

$$Y^{(e)} = f(\mathcal{P}(X^{(e)}, X; \theta_p); \theta),$$

where  $\mathcal{P}(X^{(e)}, X; \theta_p)$  denotes the feature propagation process. In order to make a fair comparison, we use the same linear link function  $f(x; \theta)$  parameterized by  $f(x; \theta) = \theta^\top x$  for all of the feature propagation processes  $\mathcal{P}(\cdot)$ , and finally feed to the cross-entropy loss function:

$$\mathcal{L} = \sum_{i=1}^m (-y_{i,0}^{(e)} \log y_{i,0}^{(e)} - y_{i,1}^{(e)} \log y_{i,1}^{(e)}) + \lambda (\|W_1\|^2 + \|W_2\|^2). \quad (21)$$

We will change the feature propagation function  $\mathcal{P}(\cdot)$  to study the performance of different types of feature propagation processes. Specifically, we design the following two feature propagation processes in the control group, and compare with edge2vec as the treatment.

**Control1.** The first type is no feature propagation, i.e. we do not expand the edge feature at all. That is,

$$\mathcal{P}(X^{(e)}, X; \theta_p) = X^{(e)}.$$

**Control2.** The second type is to only expand the edge feature by concatenating its source and target node features, that is,

$$\mathcal{P}(X^{(e)}, X; \theta_p) = \text{CONCAT}(X^{(e)}, C_s X, C_t X).$$

**Treatment (edge2vec).** The third type is to expand the edge feature by the propagation process defined in section 5, that is,

$$\mathcal{P}(X^{(e)}, X; \theta_p) = \widetilde{X}^{(e)}$$

where  $\widetilde{X}^{(e)}$  is computed by Eq. (20).

## 6.3 Results and Analyses

We plot the PR-curves<sup>6</sup> of comparison approaches in Figure 2. We could see that the result of the treatment method

<sup>6</sup>[https://en.wikipedia.org/wiki/Precision\\_and\\_recall](https://en.wikipedia.org/wiki/Precision_and_recall)

edge2vec with an appropriate  $\lambda$  performs much better than Control1 and a little better than Control2. Although the gain between treatment and Control2 is not such significant, it is in line with our expectations that feature propagation could improve the performance of prediction model.

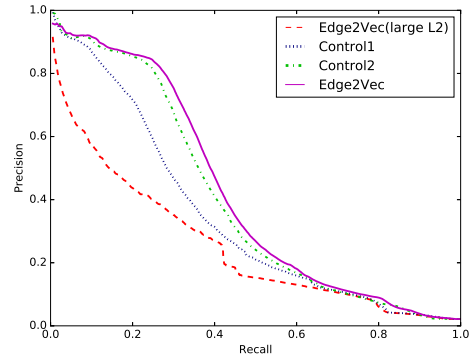


Figure 2: Precision-Recall Curve

We also analyze the potential numerical issues by testing the structure2vec method [Dai *et al.*, 2016]. For the structure2vec method, its loss function introduced a penalty parameter  $\lambda$  to constrain the value of  $W_1$  and  $W_2$ . If  $\lambda$  is set up as a small value, the numerical issue will rise up. We take four numbers of  $\lambda$   $\{10^{-3}, 10^{-4}, 10^{-5}, 10^{-6}\}$  and then test in which order of steps  $\{1\text{-order}, 2\text{-orders}, 3\text{-orders}, 4\text{-orders}, 5\text{-orders}\}$  (see into section 4.1) will lead to numerical overflow.

The following table displays the test results. We can find out that when  $\lambda$  becomes small enough ( $10^{-5}$  or  $10^{-6}$ ), the numerical overflow issue happens. However, setting a large  $\lambda$  is not a good method to handle this issue, for a large  $\lambda$  may weaken the model’s performance very sharply (see the curve of edge2vec under  $\lambda = 0.01$  in Figure 2).

Table 2: Numeric Overflow (Overflow or Not).

	1-order	2-orders	3-orders	4-orders	5-orders
$10^{-3}$	N	N	N	N	N
$10^{-4}$	N	N	N	N	N
$10^{-5}$	N	Y	Y	Y	Y
$10^{-6}$	Y	Y	Y	Y	Y

## 7 Conclusion and Future Work

In this paper, we proposed a new concept “feature propagation” and a typical way for feature propagation. We proved that convergence is a noteworthy issue for feature propagation and proposed certain conditions to guarantee its convergence. Then we revisited the two types of graph representation learning methods and found both of them have strong connections with feature propagation. Although we only revisited very limited graph representation learning methods, we provided a new perspective for understanding the essence of graph representation learning. The experiment on fraud transaction detection demonstrated the method with feature propagation could do better than the method without it. We also tested the numerical overflow issue in structure2vec. It’s a pity that we only pointed out the issue but haven’t proposed a practical way to make it. We think it is a worthy direction to explore in the future.

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