Manipulating Opinion Dynamics in Social Networks in Two Phases

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Abstract

We propose a setting for two-phase opinion dynamics in social networks, where the final opinion of a node in the first phase acts as its initial biased opinion in the second phase. In this setting, we study the problem of two camps aiming to maximize adoption of their respective opinions by strategically investing on nodes, where the effectiveness of a camp's investment on a node depends on the node's initial bias. We propose an extension of Friedkin-Johnsen model for our setting, and hence formulate the utility functions of the camps. For the non-competitive case where only one camp invests, we present a polynomial time algorithm for determining an optimal way to split the camp's budget between the two phases. For the case of competing camps, we show the existence of Nash equilibria under reasonable assumptions, and that they can be computed in polynomial time. Our main conclusion is that, if nodes attribute high weightage to their initial biases, it is advantageous to have a high investment in the first phase, so as to exploit the manipulated biases in the second phase.

1 Introduction

Studying opinion dynamics in a society is important to understand and influence elections, viral marketing, propagation of ideas and behaviors, etc. In this paper, we consider two camps who aim to maximize the adoption of their respective opinions in a social network. We consider a strict competition in the space of real-valued opinions, where one camp aims to drive the overall opinion of the network towards being positive while the other camp aims to drive it towards negative; we refer to them as good and bad camps respectively. We consider a well-accepted quantification of the overall opinion of a network: the average or equivalently, sum of opinion values of nodes [Gionis *et al.*, 2013; Grabisch *et al.*, 2018]. Hence the good and bad camps simultaneously aim to respectively maximize and minimize this sum.

The average or sum of opinion values is of relevance in several applications, e.g., fund collection where the magnitude of a node's opinion value corresponds to the amount and the sign indicates the camp towards which it is willing to contribute. Another example is when nodes are reporting agents on social media, where a node's opinion corresponds to its reported intensity of an event based on the information it receives; this information could be influenced by the camps. In such scenarios, the opinion value of a node would be a real-valued number, and the sum or average of opinion values would be the overall opinion of the network.

Social networks play a prime role in determining the opinions of constituent nodes, since nodes usually update their opinions based on the opinions of their connections [Easley and Kleinberg, 2010; Acemoglu and Ozdaglar, 2011]. This fact is exploited by camps; they determine influential nodes to invest on in the form of money, free products, convincing discussions, etc., so as to drive these nodes' opinions in their favor. Thus given a budget constraint, the strategy of a camp comprises of how much to invest on each node, in presence of a competing camp who also would invest strategically.

1.1 Motivation

In the popular model by Friedkin and Johnsen [1990, 1997], every node holds an initial bias in opinion. It could have formed owing to the node's fundamental views, experiences, mass media exposure, opinion dynamics in the past, etc. This initial bias plays an important role in determining a node's final opinion, and consequently the opinions of its neighbors and hence that of its neighbors' neighbors and so on. If nodes give significant weightage to their biases, the camps would want to manipulate these biases. This could be achieved by campaigning in multiple phases, wherein the opinion at the conclusion of a phase would act as the initial biased opinion for the next phase. Such campaigning is often used during elections and marketing, to gradually drive nodes' opinions.

The initial bias of a node often impacts the effectiveness of camps' investments on that node. For instance, if the initial bias of a node is positive, the investment made by the good camp is likely to be more effective on it than that made by the bad camp. The reasoning is similar to that of the bounded confidence model [Krause, 2000], where a node pays more attention to opinions that do not differ too much from its own opinion. Since a camp's effectiveness depends on the nodes' biases, its investment in the first phase not only manipulates the biases for getting a head start in the second phase, but also the effectiveness of its investment in the second phase.

With the possibility of campaigning in two phases, a camp could not only decide which nodes to invest on, but also how to split its available budget between the two phases.

1.2 Related Work

Several models of opinion dynamics have been proposed in the literature [Acemoglu and Ozdaglar, 2011; Lorenz, 2007], some noteworthy ones being DeGroot [DeGroot, 1974], Voter [Holley and Liggett, 1975], Friedkin-Johnsen [Friedkin and Johnsen, 1990, 1997], bounded confidence [Krause, 2000], etc. In Friedkin-Johnsen model, each node updates its opinion using a weighted combination of its initial bias and its neighbors' opinions. We generalize this model to multiple phases, while also accounting for the camps' investments.

Problems related to determining influential nodes in social networks have been extensively studied [Guille *et al.*, 2013; Easley and Kleinberg, 2010; Kempe *et al.*, 2003; Gionis *et al.*, 2013]. For instance, Yildiz *et al.* [2013] study the problem of optimal placement of stubborn nodes (whose opinion values stay unchanged) in the discrete binary opinions setting. There also have been game theoretic studies [Ghaderi and Srikant, 2014; Anagnostopoulos *et al.*, 2015; Bharathi *et al.*, 2007]. Specific to analytically tractable models such as DeGroot, there have been studies in the competitive setting to identify influential nodes and the amounts to be invested on them [Dubey *et al.*, 2006; Bimpikis *et al.*, 2016; Grabisch *et al.*, 2018]. Our work extends these studies to two phases, by identifying influential nodes in the two phases and how much they should be invested on in each phase.

There have been a few studies on adaptive selection of influential nodes for opinion diffusion in multiple phases [Seeman and Singer, 2013; Rubinstein *et al.*, 2015; Horel and Singer, 2015; Correa *et al.*, 2015; Badanidiyuru *et al.*, 2016; Tong *et al.*, 2016; Yuan and Tang, 2017]. Singer [2016] presents a survey of such adaptive methodologies. Dhamal *et al.* [2016] empirically study the problem of optimally splitting the available budget between two phases, which has been extended to multiple phases by Dhamal [2018]. While the reasoning behind using multiple phases in these studies is adaptation of strategy based on previous observations, we aim to use multiple phases for manipulating the initial biases of nodes, which requires a very different analytical treatment.

To the best of our knowledge, there has not been an analytical study on a rich model such as Friedkin-Johnsen, for opinion dynamics in two phases (not even for single camp). The most relevant to this study is our work in [Dhamal *et al.*, 2018] where, however, a camp's influence on a node is assumed to be independent of the node's bias.

1.3 Our Contributions

- We formulate the two-phase objective function under Friedkin-Johnsen model, where a node's final opinion in the first phase acts as its initial bias for the second phase, and the effectiveness of a camp's influence on the node depends on this initial bias.
- For the non-competitive case, we develop a polynomial time algorithm for determining an optimal way to split a camp's budget between the two phases.
- For the case of two competing camps, we show the existence of Nash equilibria under reasonable assumptions, and that they can be computed in polynomial time.

2 Our Model

Given a social network, let N be the set of nodes and E be the set of weighted directed edges. Let $n = |N|$. The weights could hold any sign, since the influences could be positive or negative [Altafini, 2013]. Our model can be viewed as a multiphase extension of [Dhamal *et al.*, 2017].

2.1 Parameters

As our opinion dynamics runs in two phases, most parameters have two values, one for each phase. For such parameters, we denote its value corresponding to phase p using superscript (p), where $p = 1, 2$ for first and second phase, respectively.

Initial Bias. Every node i holds an initial bias in opinion prior to the opinion dynamics process; we denote it by $v_i^0 \in \mathbb{R}$. In the multiphase setting, this acts as the initial bias for the first phase. We denote the opinion value of node i at the conclusion of phase 1 by $v_i^{(1)}$. This acts as its initial bias for phase 2. So by convention, we have $v_i^{(0)} = v_i^0$. We denote the weightage that node i attributes to its initial bias by w_{ii}^0.

Network Effect. Let w_{ij} be the weightage attributed by node i to the opinion of its connection j. Consistent with Friedkin-Johnsen model, we assume the influence on node i owing to node j in phase p, to be $w_{ij}v_j^{(p)}$. So the net influence on node i owing to all of its connections is $\sum_{j \in N} w_{ij}v_j^{(p)}$.

Weightage to Campaigning. We denote the weightage that node i attributes to the good and bad campaigning in phase p by $w_{ig}^{(p)}$ and $w_{ib}^{(p)}$, respectively. Since we consider that the initial bias of a node impacts the effectiveness of camps' investments, $v_i^{(p-1)} > 0$ would likely mean $w_{ig}^{(p)} > w_{ib}^{(p)}$. Note that w_{ii}^0 would also play a role since it quantifies the weightage given by node i to its initial bias. We hence propose a model on this line, wherein $w_{ig}^{(p)}$ is a monotone non-decreasing function of $w_{ii}^0 v_i^{(p-1)}$, and $w_{ib}^{(p)}$ is a monotone non-increasing function of $w_{ii}^0 v_i^{(p-1)}$. Let node i attribute a total of θ_i to the influence weights of the camps, that is, $w_{ig}^{(p)} + w_{ib}^{(p)} = \theta_i$. We propose the following natural model:

$$w_{ig}^{(p)} = \theta_i \left(\frac{1 + w_{ii}^0 v_i^{(p-1)}}{2} \right), \; w_{ib}^{(p)} = \theta_i \left(\frac{1 - w_{ii}^0 v_i^{(p-1)}}{2} \right) \quad (1)$$

Camp Investments. The good and bad camps attempt to directly influence the nodes so that their opinions are pulled towards being positive and negative respectively. We denote the investments made by the good and bad camps on node i in phase p by $x_i^{(p)}$ and $y_i^{(p)}$ respectively ($x_i^{(p)}, y_i^{(p)} \geq 0, \forall i \in N$ for $p = 1, 2$). Since the influence of good camp (+ve opinion) on node i in phase p would be an increasing function of both $x_i^{(p)}$ and $w_{ig}^{(p)}$, we assume the influence to be $+w_{ig}^{(p)} x_i^{(p)}$ (maintaining the linearity of Friedkin-Johnsen model). Similarly, $-w_{ib}^{(p)} y_i^{(p)}$ is the influence of bad camp ($-$ve opinion) on node i in phase p. Let k_g and k_b be the respective budgets of the good and bad camps. Hence the camps should invest in the two phases such that $\sum_{i \in N}(x_i^{(1)} + x_i^{(2)}) \leq k_g$ and $\sum_{i \in N}(y_i^{(1)} + y_i^{(2)}) \leq k_b$.

Weight Constraints. Friedkin-Johnsen model has the following condition on influence weights:

$$\forall i \in N: \ |w_{ii}^0| + \sum_{j \in N} |w_{ij}| \leq 1$$

Since the model follows an opinion update rule (as we see later), convergence is an important factor. A standard assumption for guaranteeing convergence is

$$\sum_{j \in N} |w_{ij}| < 1$$

This condition is usually enforced to ensure convergence. In our setting, this condition is natural and well suited, since we would have non-zero weights attributed to bias (w_{ii}^0).

Matrix Forms. Let $\mathbf{W}$ be the matrix consisting of weights w_{ij} for each pair of nodes (i, j). Let $\mathbf{v^0}$, $\mathbf{w^0}$, Θ, $\mathbf{w_g}^{(\mathbf{p})}$, $\mathbf{w_b}^{(\mathbf{p})}$, $\mathbf{x^{(p)}}$, $\mathbf{y^{(p)}}$, $\mathbf{v^{(p)}}$ be the vectors consisting of elements v_i^0, w_{ii}^0, θ_i, $w_{ig}^{(p)}$, $w_{ib}^{(p)}$, $x_i^{(p)}$, $y_i^{(p)}$, $v_i^{(p)}$, respectively. Let operation $\circ$ denote Hadamard vector product, i.e., $(\mathbf{a} \circ \mathbf{b})_i = a_i b_i$.

2.2 Opinion Update Rule

The update rule in Friedkin-Johnsen model is $\forall i \in N : v_i \leftarrow w_{ii}^0 v_i^0 + \sum_{j \in N} w_{ij} v_j$. Extending to multiple phases, the update rule in the p^{th} phase is $\forall i \in N : v_i^{(p)} \leftarrow w_{ii}^0 v_i^{(p-1)} + \sum_{j \in N} w_{ij} v_j^{(p)}$. Accounting for camps' investments, we get

$$\forall i \in N: \ v_i^{(p)} \leftarrow w_{ii}^0 v_i^{(p-1)} + \sum_{j \in N} w_{ij} v_j^{(p)} + w_{ig}^{(p)} x_i^{(p)} - w_{ib}^{(p)} y_i^{(p)}$$

$$\iff \mathbf{v^{(p)}} \leftarrow \mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{W}\mathbf{v^{(p)}} + \mathbf{w_g}^{(\mathbf{p})} \circ \mathbf{x^{(p)}} - \mathbf{w_b}^{(\mathbf{p})} \circ \mathbf{y^{(p)}} \quad (2)$$

In phase p, the vectors $\mathbf{x^{(p)}}, \mathbf{y^{(p)}}, \mathbf{v^{(p-1)}}$ stay unchanged; the weights $\mathbf{w_g}^{(\mathbf{p})}, \mathbf{w_b}^{(\mathbf{p})}$ (which depend on $\mathbf{v^{(p-1)}}$) and $\mathbf{w^0}$ also stay unchanged; while $\mathbf{v^{(p)}}$ gets updated. Hence, writing the update rule as recursion (with iterating integer $\tau \geq 0$):

$$\mathbf{v}_{\langle\tau\rangle}^{(\mathbf{p})} = \mathbf{W}\mathbf{v}_{\langle\tau-1\rangle}^{(\mathbf{p})} + \mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{w_g}^{(\mathbf{p})} \circ \mathbf{x^{(p)}} - \mathbf{w_b}^{(\mathbf{p})} \circ \mathbf{y^{(p)}}$$

It can be simplified as

$$\mathbf{v}_{\langle\tau\rangle}^{(\mathbf{p})} = \mathbf{W}^\tau \mathbf{v}_{\langle 0\rangle}^{(\mathbf{p})} + \Big(\sum_{\eta=0}^{\tau-1} \mathbf{W}^\eta\Big)(\mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{w_g}^{(\mathbf{p})} \circ \mathbf{x^{(p)}} - \mathbf{w_b}^{(\mathbf{p})} \circ \mathbf{y^{(p)}})$$

Now, the initial bias for phase p : $\mathbf{v}_{\langle 0\rangle}^{(\mathbf{p})} = \mathbf{v^{(p-1)}}$. Also, $\mathbf{W}$ is strictly substochastic (sum of each row strictly less than 1); its spectral radius is less than 1. So when $\tau \to \infty$, we have $\lim_{\tau \to \infty} \mathbf{W}^\tau = \mathbf{0}$ and $\lim_{\tau \to \infty} \sum_{\eta=0}^{\tau-1} \mathbf{W}^\eta = (\mathbf{I} - \mathbf{W})^{-1}$ [Grabisch *et al.*, 2018]. Hence,

$$\lim_{\tau \to \infty} \mathbf{v}_{\langle\tau\rangle}^{(\mathbf{p})} = (\mathbf{I} - \mathbf{W})^{-1}(\mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{w_g}^{(\mathbf{p})} \circ \mathbf{x^{(p)}} - \mathbf{w_b}^{(\mathbf{p})} \circ \mathbf{y^{(p)}})$$

which is a constant vector. So the dynamics in phase p converges to the steady state

$$\mathbf{v^{(p)}} = (\mathbf{I} - \mathbf{W})^{-1}(\mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{w_g}^{(\mathbf{p})} \circ \mathbf{x^{(p)}} - \mathbf{w_b}^{(\mathbf{p})} \circ \mathbf{y^{(p)}}) \quad (3)$$

2.3 Formulation of Two-Phase Objective Function

We now derive $\sum_{i \in N} v_i^{(p)}$, the sum of nodes' opinions at the end of phase $p \in \{1, 2\}$. Premultiplying (3) by $\mathbf{1}^T$ gives

$$\mathbf{1}^T \mathbf{v^{(p)}} = \mathbf{1}^T (\mathbf{I} - \mathbf{W})^{-1} (\mathbf{w^0} \circ \mathbf{v^{(p-1)}} + \mathbf{w_g} \circ \mathbf{x^{(p)}} - \mathbf{w_b} \circ \mathbf{y^{(p)}})$$

Let $\Delta = (\mathbf{I} - \mathbf{W})^{-1}$ and $\mathbf{r}^T = \mathbf{1}^T (\mathbf{I} - \mathbf{W})^{-1}$, that is, $r_i = \sum_{j \in N} \Delta_{ji}$. Since $\Delta = \sum_{\eta=0}^{\infty} \mathbf{W}^\eta$, we have that Δ_{ji} is the influence that j receives from i through walks of all possible lengths. So $r_i = \sum_{j \in N} \Delta_{ji}$ can be viewed as overall influencing power of i. Substituting these in above equation,

$$\sum_{i \in N} v_i^{(p)} = \sum_{i \in N} r_i (w_{ii}^0 v_i^{(p-1)} + w_{ig} x_i^{(p)} - w_{ib} y_i^{(p)})$$

When $p = 1$, this is the sum of opinions at the end of phase 1:

$$\sum_{i \in N} v_i^{(1)} = \sum_{i \in N} r_i (w_{ii}^0 v_i^{(0)} + w_{ig}^{(1)} x_i^{(1)} - w_{ib}^{(1)} y_i^{(1)}) \quad (4)$$

Similarly, the sum of opinion values at the end of phase 2 is

$$\sum_{j \in N} v_j^{(2)} = \sum_{j \in N} r_j (w_{jj}^0 v_j^{(1)} + w_{jg}^{(2)} x_j^{(2)} - w_{jb}^{(2)} y_j^{(2)})$$
$$= \sum_{j \in N} r_j \left(w_{jj}^0 v_j^{(1)} + \frac{\theta_j}{2}(1 + w_{jj}^0 v_j^{(1)}) x_j^{(2)} - \frac{\theta_j}{2}(1 - w_{jj}^0 v_j^{(1)}) y_j^{(2)} \right)$$
$$= \sum_{j \in N} r_j w_{jj}^0 v_j^{(1)} \left(1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)} \right) + \sum_{j \in N} r_j \frac{\theta_j}{2} (x_j^{(2)} - y_j^{(2)})$$

The first term $\sum_{j \in N} r_j w_{jj}^0 v_j^{(1)} \left(1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)}\right)$ can be obtained by premultiplying (4) by $(\mathbf{r} \circ \mathbf{w^0} \circ (\mathbf{1} + \frac{\Theta}{2} \circ \mathbf{x^{(2)}} + \frac{\Theta}{2} \circ \mathbf{y^{(2)}}))^T$ and using (1). Hence $\sum_{i \in N} v_i^{(2)}$ equals

$$\sum_{i \in N} \sum_{j \in N} \left(w_{ii}^0 v_i^0 \left(1 + \frac{\theta_i}{2} x_i^{(1)} + \frac{\theta_i}{2} y_i^{(1)}\right) + \frac{\theta_i}{2}(x_i^{(1)} - y_i^{(1)}) \right) \cdot$$
$$\cdot \left(r_j w_{jj}^0 \Delta_{ji} \left(1 + \frac{\theta_j}{2} x_j^{(2)} + \frac{\theta_j}{2} y_j^{(2)}\right) \right) + \sum_{j \in N} r_j \frac{\theta_j}{2} (x_j^{(2)} - y_j^{(2)}) \quad (5)$$

Here $(\mathbf{x^{(1)}}, \mathbf{x^{(2)}})$ is the strategy of the good camp for the two phases, and $(\mathbf{y^{(1)}}, \mathbf{y^{(2)}})$ is the strategy of the bad camp. Given an investment strategy profile $\big((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}})\big)$, let $u_g\big((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}})\big)$ be the good camp's utility and $u_b\big((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}})\big)$ be the bad camp's utility. The good camp aims to maximize the value of (5), while the bad camp aims to minimize it. So,

$$u_g\big((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}})\big) = \sum_{i \in N} v_i^{(2)}$$
$$\text{and} \quad u_b\big((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}})\big) = -\sum_{i \in N} v_i^{(2)} \quad (6)$$

with the following constraints on the investment strategies:

$$\sum_{i \in N} \left(x_i^{(1)} + x_i^{(2)}\right) \leq k_g \ , \ \sum_{i \in N} \left(y_i^{(1)} + y_i^{(2)}\right) \leq k_b$$
$$\forall i \in N : x_i^{(1)}, x_i^{(2)}, y_i^{(1)}, y_i^{(2)} \geq 0$$

The game can thus be viewed as a two-player zero-sum game, where the players determine their investment strategies $(\mathbf{x^{(1)}}, \mathbf{x^{(2)}})$ and $(\mathbf{y^{(1)}}, \mathbf{y^{(2)}})$; the good camp invests as per $\mathbf{x}^{(1)}$ in the first phase and as per $\mathbf{x}^{(2)}$ in the second phase, and the bad camp invests as per $\mathbf{y}^{(1)}$ in the first phase and as per $\mathbf{y}^{(2)}$ in the second phase. Our objective essentially is to find the Nash equilibrium strategies of the two camps.

First we consider a simplified yet interesting (and not-yet-studied-in-literature) case where budget of one of the camps is 0 (say $k_b = 0$); so effectively we have only the good camp.

3 The Non-Competitive Case

For notational simplicity, let $b_{ji} = r_j w_{jj}^0 \Delta_{ji}$, $c_i = w_{ii}^0 v_i^0$. We saw that $r_i = \sum_{j \in N} \Delta_{ji}$ indicates the influencing power of node i. Now, $b_{ji} = \Delta_{ji} w_{jj}^0 r_j$ quantifies the overall influence Δ_{ji} of i on j, which would give weightage w_{jj}^0 to its initial opinion in the next phase, and have an influencing power of r_j in the next phase. Hence b_{ji} can be interpreted as the influence of node i on the network through node j, looking one phase ahead. So $\sum_{j \in N} b_{ji}$ can be viewed as the overall influencing power of node i, looking one phase ahead. We denote $s_i = \sum_{j \in N} b_{ji}$ and identify its role in our analysis.

For non-competitive case, we have $y_i^{(1)} = y_i^{(2)} = 0, \forall i \in N$ in (5). Using above notation, we get that $\sum_{i \in N} v_i^{(2)}$ equals

$$\sum_{i \in N} \sum_{j \in N} \left(c_i + \frac{\theta_i}{2} x_i^{(1)} (c_i + 1) \right) \left(b_{ji} \left(1 + \frac{\theta_j}{2} x_j^{(2)} \right) \right) + \sum_{j \in N} r_j \frac{\theta_j}{2} x_j^{(2)} \quad (7)$$

$$= \sum_{i \in N} x_i^{(1)} \left(\frac{\theta_i}{2} (c_i + 1) \sum_{j \in N} b_{ji} \left(1 + \frac{\theta_j}{2} x_j^{(2)} \right) \right) + \sum_{i \in N} \sum_{j \in N} c_i b_{ji} \left(1 + \frac{\theta_j}{2} x_j^{(2)} \right) + \sum_{j \in N} r_j \frac{\theta_j}{2} x_j^{(2)} \quad (8)$$

$$= \sum_{j \in N} x_j^{(2)} \left(\frac{\theta_j}{2} \sum_{i \in N} b_{ji} \left(c_i + \frac{\theta_i}{2} x_i^{(1)} (c_i + 1) \right) + \frac{\theta_j}{2} r_j \right) + \sum_{i \in N} \sum_{j \in N} b_{ji} \left(c_i + \frac{\theta_i}{2} x_i^{(1)} (c_i + 1) \right) \quad (9)$$

Deducing from Equations (8) and (9) that $\sum_{i \in N} v_i^{(2)}$ is a bilinear function in $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$, we prove our next result.

Let the budget k_g be split such that $k_g^{(1)}$ and $k_g^{(2)}$ are the first and second phase investments, respectively. Our method of finding optimal $k_g^{(1)}$ and $k_g^{(2)}$ is to first search for them in the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$ and then compare the thus obtained value of $\sum_{i \in N} v_i^{(2)}$ with that corresponding to $k_g^{(1)} = k_g^{(2)} = 0$, to get the optimal value of $\sum_{i \in N} v_i^{(2)}$.

Proposition 1. *In the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$, it is optimal for good camp to exhaust entire budget ($k_g^{(1)} + k_g^{(2)} = k_g$), and to invest on at most one node in each phase.*

Proof. Given any $\mathbf{x}^{(2)}$, Expression (8) can be maximized with respect to $\mathbf{x}^{(1)}$ by allocating $k_g - \sum_{j \in N} x_j^{(2)}$ to a single node i that maximizes $\frac{\theta_i}{2}(c_i + 1) \sum_{j \in N} b_{ji} \left(1 + \frac{\theta_j}{2} x_j^{(2)} \right)$, if this value is positive. In case of multiple such nodes, one node can be chosen at random. If this value is non-positive for all nodes, it is optimal to have $\mathbf{x}^{(1)} = \mathbf{0}$. When $\mathbf{x}^{(1)} = \mathbf{0}$, Expression (9) now implies that it is optimal to allocate the entire budget k_g in second phase to a single node j that maximizes $\frac{\theta_j}{2} \left(\sum_{i \in N} b_{ji} c_i + r_j \right)$, if this value is positive. If this value is non-positive for all nodes, it is optimal to have $\mathbf{x}^{(2)} = \mathbf{0}$. This is the case where starting with an $\mathbf{x}^{(2)}$, we conclude that it is optimal to either invest $k_g - \sum_{j \in N} x_j^{(2)}$ on a single node in first phase, or invest the entire budget k_g on a single node in second phase, or invest in neither phase.

Similarly using (9), starting with a given $\mathbf{x}^{(1)}$, we can conclude that it is optimal to either invest $k_g - \sum_{j \in N} x_j^{(1)}$ on a single node in the second phase, or invest the entire k_g on a single node in the first phase, or invest in neither phase.

So starting from any $\mathbf{x}^{(1)}$ or $\mathbf{x}^{(2)}$, we can iteratively improve (need not be strictly) on the value of (7) by investing on at most one node in a given phase. Furthermore, it is suboptimal to have $k_g^{(1)} + k_g^{(2)} < k_g$ unless $k_g^{(1)} = k_g^{(2)} = 0$. □

So there exist optimal vectors $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ that maximize (7), such that $x_\alpha^{(1)} = k_g^{(1)}, x_\beta^{(2)} = k_g^{(2)}, x_{i \neq \alpha}^{(1)} = x_{j \neq \beta}^{(2)} = 0$. Now the next step is to find nodes α and β that maximize (7). By incorporating α and β in (7), we have $\sum_{i \in N} v_i^{(2)}$ equals

$$\sum_{i \neq \alpha} \sum_{j \neq \beta} c_i b_{ji} + \left(c_\alpha + \frac{\theta_\alpha}{2} k_g^{(1)} (c_\alpha + 1) \right) \left(b_{\beta\alpha} \left(1 + \frac{\theta_\beta}{2} k_g^{(2)} \right) \right) + r_\beta \frac{\theta_\beta}{2} k_g^{(2)} + \sum_{j \neq \beta} b_{j\alpha} \left(c_\alpha + \frac{\theta_\alpha}{2} k_g^{(1)} (c_\alpha + 1) \right) + \sum_{i \neq \alpha} c_i \left(b_{\beta i} \left(1 + \frac{\theta_\beta}{2} k_g^{(2)} \right) \right)$$

$$= \sum_{i \in N} \sum_{j \in N} c_i b_{ji} + \frac{\theta_\alpha}{2} k_g^{(1)} \sum_j b_{j\alpha} (c_\alpha + 1) + \frac{\theta_\beta}{2} k_g^{(2)} \Big(\sum_{i \in N} b_{\beta i} c_i + r_\beta \Big) + \frac{\theta_\alpha \theta_\beta}{4} k_g^{(1)} k_g^{(2)} b_{\beta\alpha} (c_\alpha + 1) \quad (10)$$

Now, for a given pair (α, β), we will find the optimal values of $k_g^{(1)}$ and $k_g^{(2)}$ from (10). From Proposition 1, we have $k_g^{(2)} = k_g - k_g^{(1)}$. So the expression to be maximized is

$$k_g^{(1)} \frac{\theta_\alpha}{2} \sum_{j \in N} b_{j\alpha} (c_\alpha + 1) + (k_g - k_g^{(1)}) \frac{\theta_\beta}{2} \Big(\sum_{i \in N} b_{\beta i} c_i + r_\beta \Big) + k_g^{(1)} (k_g - k_g^{(1)}) \frac{\theta_\alpha \theta_\beta}{4} b_{\beta\alpha} (c_\alpha + 1) + \sum_{i \in N} \sum_{j \in N} c_i b_{ji}$$

Equating its first derivative w.r.t. $k_g^{(1)}$ to zero, we get

$$k_g^{(1)} = \frac{k_g}{2} + \frac{\sum_{j \in N} b_{j\alpha}}{\theta_\beta b_{\beta\alpha}} - \frac{\sum_{i \in N} b_{\beta i} c_i + r_\beta}{\theta_\alpha b_{\beta\alpha} (c_\alpha + 1)}$$

A valid value of $k_g^{(1)}$ can be obtained only if the denominators in above expression are non-zero. However, a zero denominator would mean that Expression (10) is linear, resulting in only two possibilities of $k_g^{(1)}$, namely, 0 or k_g. Also, if the second derivative with respect to $k_g^{(1)}$ is positive, that is, $-\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha} (w_{\alpha\alpha}^0 v_\alpha^0 + 1) > 0$, optimal $k_g^{(1)}$ is either 0 or k_g. If the second derivative with respect to $k_g^{(1)}$ is negative: $-\theta_\alpha \theta_\beta b_{\beta\alpha} (c_\alpha + 1) = -\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha} (w_{\alpha\alpha}^0 v_\alpha^0 + 1) < 0$, and since $k_g^{(1)}$ is bounded in $[0, k_g]$, optimal $k_g^{(1)}$ for pair (α, β) is (since $c_i = w_{ii}^0 v_i^0$, $b_{ji} = r_j w_{jj}^0 \Delta_{ji}$, $s_i = \sum_{j \in N} b_{ji}$):

$$\min \left\{ \max \left\{ \frac{k_g}{2} + \frac{s_\alpha}{\theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}} - \frac{1 + w_{\beta\beta}^0 \sum_{i \in N} \Delta_{\beta i} w_{ii}^0 v_i^0}{\theta_\alpha w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0)}, 0 \right\}, k_g \right\} \quad (11)$$

and the corresponding optimal value of $k_g^{(2)}$ is

$$\min \left\{ \max \left\{ \frac{k_g}{2} - \frac{s_\alpha}{\theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}} + \frac{1 + w_{\beta\beta}^0 \sum_{i \in N} \Delta_{\beta i} w_{ii}^0 v_i^0}{\theta_\alpha w_{\beta\beta}^0 \Delta_{\beta\alpha} (1 + w_{\alpha\alpha}^0 v_\alpha^0)}, 0 \right\}, k_g \right\} \quad (12)$$

When we assumed $k_g^{(1)}$ and $k_g^{(2)}$ to be fixed, we had to iterate through all (α, β) pairs to determine the one that gives the optimal value of Expression (10). Now, whenever we look at an (α, β) pair, we can determine the corresponding optimal values of $k_g^{(1)}$ and $k_g^{(2)}$ using (11) and (12), and hence determine the value of Expression (10) by plugging in the optimal $k_g^{(1)}$ and $k_g^{(2)}$ and that (α, β) pair. The optimal pair (α, β) can thus be obtained as the pair that maximizes (10).

Above analysis holds when $k_g^{(1)} + k_g^{(2)} = k_g$. From Proposition 1, we need to consider one more possibility that $k_g^{(1)} = k_g^{(2)} = 0$, which gives a constant value $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$ for Expression (10). Let $(0, 0)$ correspond to this additional possibility. It is hence optimal to invest $k_g^{(1)}$ (obtained using (11)) on node α in the first phase and $k_g^{(2)}$ (obtained using (12)) on node β in the second phase, subject to it giving a value greater than $\sum_{i \in N} \sum_{j \in N} c_i b_{ji}$ to Expression (10).

Since we iterate through $(n^2 + 1)$ possibilities (namely, $(\alpha, \beta) \in N \times N \cup \{(0, 0)\}$), the above procedure gives a polynomial time algorithm for determining the optimal budget split and the optimal investments on nodes in two phases.

Remark 1. *For non-negative values of parameters, (11) indicates that for a given (α, β) pair, the good camp would want to invest more in the first phase for a higher s_α. This is intuitive from our understanding of s_α being viewed as the influencing power of node α looking one phase ahead. Similarly, (12) indicates that it would want to invest more in second phase for a higher r_β, since r_β can be viewed as the influencing power of node β in the immediate phase. Also, (11) and (12) indicate that a higher θ_α drives the camp to invest in first phase and a higher θ_β drives it to invest in second phase. Since w_{ig} is an increasing function of θ_i, this implicitly means that a node with a higher w_{ig} drives the good camp to invest in the phase in which that node is selected. Further, we illustrate the role of w_{ii}^0 using simulations in Section 5.*

4 The Case of Competing Camps

Similar to Proposition 1, we can show that $\sum_{i \in N} v_i^{(2)}$ is a multilinear function in this case also, since it can be written as a linear function in $\mathbf{x^{(1)}}, \mathbf{y^{(1)}}, \mathbf{x^{(2)}}, \mathbf{y^{(2)}}$ separately. The following proposition can be hence proved on similar lines.

Proposition 2. *In the search space $k_g^{(1)} + k_g^{(2)} \in (0, k_g]$, it is optimal for good camp to have $k_g^{(1)} + k_g^{(2)} = k_g$, and invest on at most one node in each phase. In the search space $k_b^{(1)} + k_b^{(2)} \in (0, k_b]$, it is optimal for bad camp to have $k_b^{(1)} + k_b^{(2)} = k_b$, and invest on at most one node in each phase.*

Denoting $w_{ii}^0 v_i^0 = c_i$, $r_j w_{jj}^0 \Delta_{ji} = b_{ji}$, $s_i = \sum_{j \in N} b_{ji}$ as before, from Expression (5), $\sum_{i \in N} v_i^{(2)}$ can be expanded as

$$\sum_{i\in N}\sum_{j\in N} c_i b_{ji} + \sum_{j\in N} x_j^{(2)} \frac{\theta_j}{2}\Big(\sum_{i\in N} c_i b_{ji} + r_j\Big) + \sum_{j\in N} y_j^{(2)} \frac{\theta_j}{2}\Big(\sum_{i\in N} c_i b_{ji} - r_j\Big)$$
$$+ \sum_{i\in N} x_i^{(1)} \frac{\theta_i}{2}(1+c_i)\Big(s_i + \sum_{j\in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j\in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji}\Big)$$
$$- \sum_{i\in N} y_i^{(1)} \frac{\theta_i}{2}(1-c_i)\Big(s_i + \sum_{j\in N} x_j^{(2)} \frac{\theta_j}{2} b_{ji} + \sum_{j\in N} y_j^{(2)} \frac{\theta_j}{2} b_{ji}\Big) \qquad (13)$$

From Proposition 2, there exist optimal vectors $\mathbf{x^{(1)}}, \mathbf{x^{(2)}}$ for good camp and optimal vectors $\mathbf{y^{(1)}}, \mathbf{y^{(2)}}$ for bad camp, such that $x_\alpha^{(1)} = k_g^{(1)}, x_\beta^{(2)} = k_g^{(2)}, y_\gamma^{(1)} = k_b^{(1)}, y_\delta^{(2)} = k_b^{(2)}$, and $x_{i\neq\alpha}^{(1)} = x_{j\neq\beta}^{(2)} = y_{i\neq\gamma}^{(1)} = y_{j\neq\delta}^{(2)} = 0$. Assuming such profile of nodes $((\alpha, \beta), (\gamma, \delta))$, we first find $((x_\alpha^{(1)}, x_\beta^{(2)}), (y_\gamma^{(1)}, y_\delta^{(2)}))$, or equivalently, the optimal $((k_g^{(1)}, k_g^{(2)}), (k_b^{(1)}, k_b^{(2)}))$ corresponding to such a profile. By incorporating $((\alpha, \beta), (\gamma, \delta))$, Expression (13) for $\sum_{i \in N} v_i^{(2)}$ simplifies to

$$\sum_{i\in N}\sum_{j\in N} c_i b_{ji} + k_g^{(2)} \frac{\theta_\beta}{2}\Big(\sum_{i\in N} c_i b_{\beta i} + r_\beta\Big) + k_b^{(2)} \frac{\theta_\delta}{2}\Big(\sum_{i\in N} c_i b_{\delta i} - r_\delta\Big)$$
$$+ k_g^{(1)} \frac{\theta_\alpha}{2}(1 + c_\alpha)\Big(s_\alpha + k_g^{(2)} \frac{\theta_\beta}{2} b_{\beta\alpha} + k_b^{(2)} \frac{\theta_\delta}{2} b_{\delta\alpha}\Big)$$
$$- k_b^{(1)} \frac{\theta_\gamma}{2}(1 - c_\gamma)\Big(s_\gamma + k_g^{(2)} \frac{\theta_\beta}{2} b_{\beta\gamma} + k_b^{(2)} \frac{\theta_\delta}{2} b_{\delta\gamma}\Big) \quad (14)$$

First, we consider the case when $k_g^{(1)} + k_g^{(2)} = k_g$ and $k_b^{(1)} + k_b^{(2)} = k_b$. Now, for a given profile of nodes $((\alpha, \beta), (\gamma, \delta))$, we will find the optimal values of $k_g^{(1)}, k_g^{(2)}, k_b^{(1)}, k_b^{(2)}$. In this case, we have $k_g^{(2)} = k_g - k_g^{(1)}$ and $k_b^{(2)} = k_b - k_b^{(1)}$. Substituting this in (14) and simultaneously solving $\frac{\partial \sum_{i \in N} v_i^{(2)}}{\partial k_g^{(1)}} = 0$ and $\frac{\partial \sum_{i \in N} v_i^{(2)}}{\partial k_b^{(1)}} = 0$, we get $k_g^{(1)}$ equal to (letting $A = \theta_\gamma \theta_\delta (1 - c_\gamma) b_{\delta\gamma}$ and $B = \frac{1}{2}\left(\theta_\alpha \theta_\delta (1 + c_\alpha) b_{\delta\alpha} - \theta_\gamma \theta_\beta (1 - c_\gamma) b_{\beta\gamma}\right)$):

$$\frac{1}{B^2 + \theta_\alpha\theta_\beta(1+c_\alpha)b_{\beta\alpha}A}\Big[s_\alpha\theta_\alpha(1+c_\alpha)A - r_\beta\theta_\beta A - s_\gamma\theta_\gamma(1-c_\gamma)B + r_\delta\theta_\delta B$$
$$+k_g\Big(\frac{\theta_\alpha\theta_\beta}{2}(1 + c_\alpha)b_{\beta\alpha}A - \frac{\theta_\gamma\theta_\beta}{2}(1 - c_\gamma)b_{\beta\gamma}B\Big)$$
$$+k_b\Big(\frac{\theta_\alpha\theta_\delta}{2}(1 + c_\alpha)b_{\delta\alpha}A - \frac{\theta_\gamma\theta_\delta}{2}(1 - c_\gamma)b_{\delta\gamma}B\Big)$$
$$-\theta_\beta A\sum_{i\in N} c_i b_{\beta i} - \theta_\delta B\sum_{i\in N} c_i b_{\delta i}\Big]$$

We can similarly obtain $k_b^{(1)}$. If second derivative w.r.t. $k_g^{(1)}$, i.e., $-\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}(1 + w_{\alpha\alpha}^0 v_\alpha^0) < 0$ and that w.r.t. $k_b^{(1)}$, i.e., $\theta_\gamma \theta_\delta r_\delta w_{\delta\delta}^0 \Delta_{\delta\gamma}(1 - w_{\gamma\gamma}^0 v_\gamma^0) > 0$, and the obtained solution is such that $k_g^{(1)} \in [0, k_g]$ and $k_b^{(1)} \in [0, k_b]$, then neither the good camp can change $k_g^{(1)}$ to increase $\sum_{i \in N} v_i^{(2)}$, nor the bad camp can change $k_b^{(1)}$ to decrease $\sum_{i \in N} v_i^{(2)}$. So we can effectively write $u_g((\mathbf{x^{(1)}}, \mathbf{x^{(2)}}), (\mathbf{y^{(1)}}, \mathbf{y^{(2)}}))$ as $u_g((\alpha, \beta), (\gamma, \delta))$, where $u_g((\alpha, \beta), (\gamma, \delta))$ is the value of $\sum_{i \in N} v_i^{(2)}$, which corresponds to the strategy profile where good camp invests on nodes (α, β) with optimal budget split $(k_g^{(1)}, k_g^{(2)})$, and bad camp invests on nodes (γ, δ) with optimal budget split $(k_b^{(1)}, k_b^{(2)})$.

For general case where the solution $k_g^{(1)}, k_b^{(1)}$ obtained above may not satisfy $k_g^{(1)} \in [0, k_g]$ and $k_b^{(1)} \in [0, k_b]$, we make a practically reasonable assumption so as to determine $u_g((\alpha, \beta), (\gamma, \delta))$. It is easy to show that if $w_{ij} \geq 0, \forall (i, j)$, then $\Delta_{ij} \geq 0, \forall (i, j)$ and $r_i \geq 1, \forall i \in N$. So if we assume $w_{ij} \geq 0, \forall (i, j)$ and $w_{ii}^0 \geq 0, \theta_i \geq 0, v_i^0 \in [-1, 1], \forall i \in N$, we would have that $-\theta_\alpha \theta_\beta r_\beta w_{\beta\beta}^0 \Delta_{\beta\alpha}(1 + w_{\alpha\alpha}^0 v_\alpha^0) \leq 0$ and

$\theta_\gamma \theta_\delta r_\delta w^0_{\delta\delta} \Delta_{\delta\gamma}(1 - w^0_{\gamma\gamma} v^0_\gamma) \geq 0$. That is, we would have $\sum_{i \in N} v_i^{(2)}$ to be a convex-concave function, which is concave w.r.t. $k_g^{(1)}$ and convex w.r.t. $k_b^{(1)}$. So in the domain $([0, k_g], [0, k_b])$, we can find a $(k_g^{(1)}, k_b^{(1)})$ such that, neither the good camp can change $k_g^{(1)}$ to increase $\sum_{i \in N} v_i^{(2)}$, nor the bad camp can change $k_b^{(1)}$ to decrease $\sum_{i \in N} v_i^{(2)}$ [Boyd and Vandenberghe, 2004; Arrow *et al.*, 1958]. So we can assign this value $\sum_{i \in N} v_i^{(2)}$ to $u_g((\alpha, \beta), (\gamma, \delta))$.

Thus using above technique, we obtain $u_g((\alpha, \beta), (\gamma, \delta))$ for all profiles of nodes $((\alpha, \beta), (\gamma, \delta))$ when $k_g^{(1)} + k_g^{(2)} = k_g$ and $k_b^{(1)} + k_b^{(2)} = k_b$. From Proposition 2, the only other cases to consider are $k_b^{(1)} = k_b^{(2)} = 0$ and $k_g^{(1)} = k_g^{(2)} = 0$. Let the profile $((\alpha, \beta), (0, 0))$ correspond to $k_b^{(1)} = k_b^{(2)} = 0$. Note that when $k_b^{(1)} = k_b^{(2)} = 0$, it reduces to non-competitive case with only good camp (Section 3); the value of $\sum_{i \in N} v_i^{(2)}$ for an (α, β) pair can hence be assigned to $u_g((\alpha, \beta), (0, 0))$. Thus we can obtain $u_g((\alpha, \beta), (\gamma, \delta))$ for all profiles of nodes $((\alpha, \beta), (0, 0))$. Similarly, we can obtain $u_g((\alpha, \beta), (\gamma, \delta))$ for all profiles of nodes $((0, 0), (\gamma, \delta))$. And from Equation (14), $u_g((0, 0), (0, 0)) = \sum_{i \in N} \sum_{j \in N} c_i b_{ji}$.

So we have that the good camp has $(n^2 + 1)$ possible pure strategies to choose from, namely, $(\alpha, \beta) \in N \times N \cup \{(0, 0)\}$. Similarly, the bad camp has $(n^2 + 1)$ possible pure strategies to choose from, namely, $(\gamma, \delta) \in N \times N \cup \{(0, 0)\}$. We thus have a two-player zero-sum game, for which the utilities of the players can be computed for each strategy profile $((\alpha, \beta), (\gamma, \delta))$ as explained above. Though we cannot ensure the existence of a pure strategy Nash equilibrium, the finiteness of the number of strategies ensures the existence of a mixed strategy Nash equilibrium. Further, owing to it being a two-player zero-sum game, the Nash equilibria can be found efficiently by solving a linear program [Osborne, 2004].

Summarizing, under practically reasonable assumptions ($w_{ij} \geq 0, \forall (i, j)$ and $w^0_{ii} \geq 0, \theta_i \geq 0, v^0_i \in [-1, 1], \forall i \in N$), we transformed the problem into a two-player zero-sum game with each player having $(n^2 + 1)$ pure strategies, and showed how the players' utilities can be computed for each strategy profile. We thus deduced the existence of Nash equilibria and that they can be found efficiently using linear programming.

5 Simulations and Results

For determining implications of our analytical results on real-world networks, we conducted simulations on NetHEPT (15,233 nodes, 31,376 edges): a dataset widely used for experimental justifications in the literature on opinion diffusion [Kempe *et al.*, 2003; Chen *et al.*, 2009, 2010]. We present our results for the non-competitive case. We point that though we presented a polynomial time algorithm for two camps, which is of theoretical interest, it is computationally expensive to run on networks larger than a few hundred nodes. It is hence worth exploring the possibility of a more efficient algorithm.

Setup. We assume the value of w^0_{ii} to be same for all nodes, in order to study the effect of this value. The range of values we consider for w^0_{ii} is $\{0, 0.05, \ldots, 0.95\}$. Let $N(i)$ be the set of i's neighbors. We model the influence weights as follows. For a given w^0_{ii}, we consider $\theta_i > 0$ and $w_{ij} \geq 0, \forall j \in N(i)$ such that the weights for any node i sum to 1, i.e., $\theta_i + \sum_{j \in N(i)} w_{ij} = 1 - w^0_{ii}$. For different values of w^0_{ii}, the values of θ_i and w_{ij} are scaled proportional to $(1 - w^0_{ii})$. We consider $v^0_i = 0, \forall i \in N$ to start with a neutral network, for reliably studying the effects of critical parameters r_i, w^0_{ii}, and hence also s_j. So, $w^{(1)}_{ig} = \frac{\theta_i}{2}$ and $w^{(2)}_{ig} = \frac{\theta_i}{2}(1 + w^0_{ii} v^{(1)}_i)$.

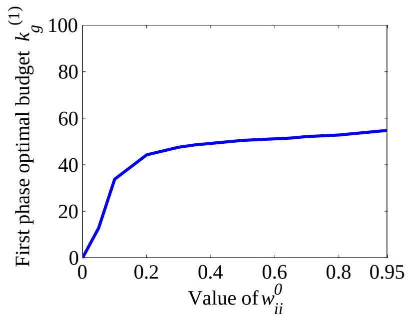

Figure 1: Illustration of the effect of w^0_{ii} (NetHEPT) with $k_g = 100$

Results. Figure 1 presents the optimal budget that should be allotted for first phase as a function of w^0_{ii}. In our simulations, the optimal values obtained are such that $k_g^{(2)} = k_g - k_g^{(1)}$. For low values of w^0_{ii}, the optimal strategy is to invest almost entirely in second phase. This is because the effect of first phase diminishes in second phase when w^0_{ii} is low. Remark 1 states that high s_j value (influencing power of j looking one phase ahead) would attract more investment in first phase. The value $s_j = \sum_{i \in N} r_i w^0_{ii} \Delta_{ij}$ would be significant only if j influences nodes i with significant values of w^0_{ii}. With low w^0_{ii}, we are less likely to have node j with high value of s_j since it requires it to be influential towards significant number of nodes i with significant values of w^0_{ii}. Hence allotting a significant budget for first phase would be advantageous only if we have nodes with significant value of w^0_{ii}.

We considered $v^0_i = 0, \forall i \in N$ in above simulations, however, we also observed the effects of $v^0_i \neq 0$ (initially biased network); the nature of plot remains similar to Figure 1 with subtle differences. If v^0_i's are mostly positive, the good camp invests less in first phase, mainly because v^0_i's already give a good head start to have a healthy value at the end of first phase (which is bias for second phase), and the budget could rather be invested in second phase. If v^0_i's are mostly negative, it invests more in first phase to nullify the initial disadvantage.

6 Conclusion

Using Friedkin-Johnsen model, we proposed a framework for two-phase investment on nodes in a social network, where a node's opinion in first phase acts as its bias in second phase, and the effectiveness of a camp's investment depends on this bias. For one investing camp, we derived polynomial time algorithm for determining optimal budget split between two phases. Our simulations quantified the impact of the weightage that nodes attribute to their biases; a high weightage necessitated high investment in first phase, so that the manipulated biases could be harnessed in second phase. For competing camps, we showed existence of Nash equilibria under reasonable assumptions ($w_{ij} \geq 0, \forall (i, j)$ and $w^0_{ii} \geq 0, \theta_i \geq 0, v^0_i \in [-1,1], \forall i \in N$) and their polynomial time computability.

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